# HomeWork 8 : The Bose-Einstien condensation

## 1. Bose condensate critical temperature

**a)** Using the Bose-Einstein distribution, the total number of particles in a box is

$$N = \sum_k \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1}.$$

In the continuum limit, we replace $\sum_k$ by $\int \frac{V}{(2\pi)^3} d^3k$, and introducing the density of states $g(\varepsilon)$ we get the density :

$$n = \int_0^\infty \frac{1}{e^{\beta(\varepsilon - \mu)} - 1} g(\varepsilon) d\varepsilon$$

we can see this equation as an implicit definition of the chemical potential as a function of $T$ and $n$. Rewrite (evaluate $g(\varepsilon)$ explicitly !) $n$ in terms of dimensionless variables $z = e^{\beta\mu}$ (called the fugacity) and $x = \beta\varepsilon$. You should find that

$$n \propto \int_0^\infty \frac{ze^{-x}}{1 - ze^{-x}} x^{1/2} dx$$

To calculate this integral we can expand

$$\frac{ze^{-x}}{1 - ze^{-x}} = \sum_{p=1}^{\infty} z^p e^{-px}$$

Prove that

$$n = \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2} g_{3/2}(z)$$

For this you will certainly need the following results :

$$\Gamma(t) = \int_0^\infty z^{t-1} e^{-y} dy$$

with the value $\Gamma(3/2) = \sqrt{\pi}/2$, and

$$g_{3/2} = \sum_{p=1}^{\infty} \frac{z^p}{p^{3/2}}$$

Invert this equation and calculate the critical temperature $T_c$ where $\mu = 0$. Comment.

## 2. Number of particles in the condensate

*NB : In this exercice, the notations will be taken from the lecture.*

The goal is to determine the temperature dependence of the number of particles in the condensate ($k = 0$), for a contact potential $V_k = \lambda$.

**a)** Proceed by first calculating the thermodynamic expectation value of the particle number operator $N = \sum_k a_k^\dagger a_k$. Rewrite it in terms of the Bogoliubov operators $\alpha_k$ and switch to the continuum limit.

*Result* :

$$N = N_0(T) + 2\frac{(mn\lambda)^{3/2}}{\pi^2}\left(\frac{1}{6} + U_1(\gamma)\right),$$

where $\gamma = \beta k_0^2/2m$, $k_0^2 = 4mn\lambda$, $\beta = 1/k_B T$, and

$$U_n(\gamma) = \int_0^\infty dy \frac{x^n}{e^{\gamma y} - 1}$$

with $y = x\sqrt{x^2+1}$

**b)** Show that for low temperature the depletion of the condensate increases quadratically with the temperature

$$\frac{N_0(T)}{V} = \frac{N_0(0)}{V} - \frac{m}{12c}(k_B T)^2$$

where $c = \sqrt{\frac{n\lambda}{m}}$.